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QUADRATIC EXTREMUM PRINCIPLES FOR FINITE STRAIN RIGID–VISCOPLASTIC DYNAMICS OF CONTINUA

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Abstract—Two quadratic extremum principles for finite strains of rigid–viscoplastic continua with piecewise linear constitutive laws are derived from and formulated using both the Eulerian and Lagrangian descriptions of the continuum motion. The principles generalize to finite strains two previous extremum properties of the dynamic solution of rigid-plastic small-strain, large displacement bodies found by Capurso. Stolarsky and Belytschko's recent non-quadratic theorem is shown to be a special form of one of the two theorems when the functional dependence from a variable is implicitly expressed.

1. INTRODUCTION

The problem of determining the large displacement dynamic response of a body under a given high intensity loading has found, on the widely accepted assumption that the elastic strains are small as compared to total strains, a fundamental hypothesis for the development of a number of methods and numerical techniques forming an important part of the more recent developments the plastic body theory.

In this context Tamuzh's[1] extremum principle, which gives the acceleration field of a rigid-plastic solid at a given time, the velocity field as well as body and surface forces being known, was extended by Capurso[2] with two extremum principles to the large displacement (small strain) case in the presence of a rigid–viscoplastic solid with a piecewise-linear yield surface. In these extensions the functionals to optimize involve the acceleration and stress field, or the acceleration and plastic multiplier acceleration field, respectively.

A corresponding quadratic programming formulation was also derived by the same author in a finite element context.

More recently, Stolarsky and Belytschko[3] extended Tamuzh's principle to the more general case of large strains with a smooth yield surface and presented the results of an application to the particular case of rod structures, using a numerical procedure to determine the rigid-plastic response by solving a series of quadratic programming problems.

In this paper Capurso's principles are extended to cover the general case of finite strains and the two corresponding principles obtained involve the same pair of functions. They are both derived by the Eulerian and Lagrangian descriptions of the deformations (kinematic description) and of stresses (kinetic description).

In Section 2 the general formulation of the constitutive laws of rigid–plastic materials on the assumption of a piecewise-linear yield surface and finite strains, is given for both descriptions considering the need to refer to objective quantities in order to provide a description independent of the reference system.

Section 3 is concerned with the formulation of the problem using both the Lagrangian and Eulerian approaches, while in Section 4 extremum principles are derived.

Finally, Section 5 deals with the particularization of the new principles to the previous ones by Capurso (large displacements with small strains) and to the derivation of Stolarsky and Belytschko's formulation.

2. CONSTITUTIVE LAWS

2.1. *Eulerian approach*

Reference is made in the following to the Eulerian description of the kinetics and kinematics. In a Cartesian orthogonal reference system let x_i be the current position occupied by a particle at time t and let a_i be its original position at time $t = 0$. The components u_i of the finite displacement vector of the particle are then given by

$$u_i = x_i - a_i. \tag{1}$$

In this context the strain field will be described by the Almansi strain tensor defined by

$$e_{ij} = \tfrac{1}{2}(u_{i/j} + u_{j/i} - u_{k/i}u_{k/j}). \tag{2}$$

The stress field will be described in the following through the Cauchy stress tensor σ_{ij} the components of which refer to the strained state of the body. Besides, the assumption is made of rigid–viscoplastic hardening and associated material behaviour with yield domain, delimited, in the stress space σ_{ij}, by n planes

$$\varphi_\alpha = N^\sigma_{\alpha ik}\sigma_{ik} - k_\alpha \leqslant 0 \quad (\alpha = 1, \ldots, n) \tag{3}$$

where $N^\sigma_{\alpha ik}$ are the n outward unit vectors normal to the n yield planes, and

$$k_\alpha = r_\alpha + \sum_\beta H_{\alpha\beta}\dot{\lambda}_\beta \tag{4}$$

where r_α defines the yield polyhedron plane distances from the origin in the original (virgin) state of the material, the interaction coefficients $H_{\alpha\beta}$ define the hardening rule, and $\dot{\lambda}_\beta$ denotes the plastic multiplier rates. In what follows r_α, $H_{\alpha\beta}$ are assumed to be independent of $\dot{\lambda}$ and matrix $H_{\alpha\beta}$ to be symmetric definite positive.

In order to ensure the invariance of the yield function with respect to any rigid body motion of the material particle, the yield condition, eqn (3), must be expressed as a function of the corotational stress tensor

$$\hat{\sigma}_{ij} = R_{hi}\sigma_{hk}R_{kj} \tag{5}$$

where R_{hi}, defining the rigid body motion of the particle, is obtained by decomposing the deformation gradient

$$F_{ij} = \frac{\partial x_i}{\partial a_j} = \delta_{ij} + \frac{\partial u_i}{\partial a_j} \tag{6}$$

into the product of the rotation tensor R_{ik} and a (right) stretch tensor U_{kj} in the form

$$F_{ij} = R_{ik}U_{kj} \tag{7}$$

being

$$R_{ik}R_{jk} = \delta_{ij} \quad \text{and} \quad R_{ki}R_{kj} = \delta_{ij}. \tag{8}$$

Then the yield condition, eqn (3), becomes

$$\varphi_\alpha = \hat{N}_{\alpha ij}\hat{\sigma}_{ij} - k_\alpha \leqslant 0 \tag{3'}$$

where

$$\hat{N}_{\alpha ij} = R_{hi} N^{\sigma}_{\alpha hk} R_{kj} \tag{9}$$

is invariant in the corotational reference system, and

$$N^{\sigma}_{\alpha ij} = R_{ih} \hat{N}_{\alpha hk} R_{jk}. \tag{10}$$

In order to describe the associated flow rule the Almansi strain tensor is used as a measure of deformations. However, the rate

$$\dot{e}_{ij} = \tfrac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i}) - (e_{ik} L_{kj} + L_{ki} e_{kj}) \tag{11}$$

is not conjugate to Cauchy stress (their scalar product does not give work) and the rate

$$D_{ij} = \tfrac{1}{2}(\dot{u}_{i;j} + \dot{u}_{j;i}) \tag{12}$$

(called "rate of deformation tensor") will be used. Then, through eqn (3), the associated flow rule becomes

$$D_{ij} = \sum_{\alpha} \dot{\lambda}_{\alpha} \frac{\partial \varphi_{\alpha}}{\partial \sigma_{ij}} = \sum_{\alpha} \dot{\lambda}_{\alpha} N^{\sigma}_{\alpha ij} \tag{13}$$

under the constraints

$$\varphi_{\alpha} \leqslant 0, \ \dot{\lambda}_{\alpha} \geqslant 0, \ \varphi_{\alpha} \dot{\lambda}_{\alpha} = 0. \tag{14}$$

Limitation to the acceleration field $\ddot{e}_{ij}$ must then be derived from eqns (13) and (14) equally involving objective quantities. However, from the material derivative of eqn (13), the following is obtained:

$$\dot{D}_{ij} = \sum_{\alpha} \ddot{\lambda}_{\alpha} \frac{\partial \varphi_{\alpha}}{\partial \sigma_{ij}} + \sum_{\alpha} \dot{\lambda}_{\alpha} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \varphi_{\alpha}}{\partial \sigma_{ij}}\right) \tag{15}$$

where

$$\dot{D}_{ij} = \tfrac{1}{2}(\ddot{u}_{i,j} + \ddot{u}_{j,i}) - \tfrac{1}{2}(\dot{u}_{k;j}\dot{u}_{i,k} + \dot{u}_{k;i}\dot{u}_{j;k}) \tag{16}$$

can be easily shown to be frame dependent. In order that objective quantities appear in the first member of eqn (15) the following relation can be used[4]:

$$\dot{D}_{ij} = D^{0}_{ij} + W_{ik} D_{kj} - D_{ik} W_{kj} \tag{17}$$

where $W_{ij} = \tfrac{1}{2}(\dot{u}_{i;j} - \dot{u}_{j;i})$ is the spin tensor and

$$D^{0}_{ij} = \dot{D}_{ij} - W_{ik} D_{kj} + D_{ik} W_{kj} = \tfrac{1}{2}(\ddot{u}_{i;j} + \ddot{u}_{j;i}) - \tfrac{1}{2}(\dot{u}_{i,k}\dot{u}_{k;j} + \dot{u}_{k;i}\dot{u}_{j;k} + \dot{u}_{i;k}\dot{u}_{j;k} - \dot{u}_{k;i}\dot{u}_{k;j}) \tag{18}$$

(referred to as the "corotational derivative of D_{ij}") may be easily proved to be frame independent. As a result eqn (15) becomes

$$D^{0}_{ij} = \sum_{\alpha} \ddot{\lambda}_{\alpha} \frac{\partial \varphi_{\alpha}}{\partial \sigma_{ij}} + \sum_{\alpha} \dot{\lambda}_{\alpha} \left[\frac{\mathrm{d}}{\mathrm{d}t}(N^{\sigma}_{\alpha ij}) - W_{ik} N^{\sigma}_{\alpha kj} + N^{\sigma}_{\alpha ik} W_{kj}\right] \tag{19}$$

where the plastic multiplier accelerations $\ddot{\lambda}_{\alpha}$ are subjected to the constraints

$$\begin{aligned} \ddot{\lambda}_\alpha &= 0 \quad \text{if } \varphi_\alpha < 0 \text{ and } \dot{\lambda}_\alpha = 0 \\ \ddot{\lambda}_\alpha &\geq 0 \quad \text{if } \varphi_\alpha = 0 \text{ and } \dot{\lambda}_\alpha = 0 \\ \ddot{\lambda}_\alpha &\text{ free} \quad \text{if } \varphi_\alpha = 0 \text{ and } \dot{\lambda}_\alpha > 0 \end{aligned} \tag{20}$$

which may be expressed as follows in a more compact form:

$$\begin{aligned} &\varphi_\alpha \ddot{\lambda}_\alpha = 0 \text{ and } \ddot{\lambda}_\alpha \geq 0 \text{ if } \varphi_\alpha \leq 0 \text{ and } \dot{\lambda}_\alpha = 0 \ (V_R) \\ &\varphi_\alpha \ddot{\lambda}_\alpha = 0 \text{ and } \ddot{\lambda}_\alpha \text{ free if } \varphi_\alpha = 0 \text{ and } \dot{\lambda}_\alpha > 0 \ (V_P). \end{aligned} \tag{21}$$

In eqn (19) the second term of the second member always vanishes. In fact, when the material point lies in a rigid region, $\dot{\lambda}_\alpha = 0$; besides, considering that[4]

$$\frac{\mathrm{d}}{\mathrm{d}t}(R_{ij}) = \dot{R}_{ij} = W_{ik}R_{kj} \tag{22}$$

we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(N^\sigma_{\alpha ij}) &= \frac{\mathrm{d}}{\mathrm{d}t}(R_{ih}\hat{N}_{\alpha hk}R_{jk}) = W_{is}R_{sh}\hat{N}_{\alpha hk}R_{jk} + R_{ih}\hat{N}_{\alpha hk}W_{jr}R_{rk} \\ &= W_{is}N^\sigma_{\alpha sj} + N^\sigma_{\alpha ir}W_{jr} = W_{is}N^\sigma_{\alpha sj} - N^\sigma_{\alpha ir}W_{rj} \end{aligned} \tag{23}$$

thus, in the plastic region too (where $\dot{\lambda}_\alpha > 0$) the second term of the second member of eqn (19) vanishes. This implies that the plastic multiplier accelerations are always governed by the condition

$$D^0_{ij} = \sum_\alpha \ddot{\lambda}_\alpha \frac{\partial \varphi_\alpha}{\partial \sigma_{ij}} = \sum_\alpha \ddot{\lambda}_\alpha N^\sigma_{\alpha ij} \tag{24}$$

stating the normality of the corotational derivative of the rate of the deformation tensor to the field surface.

In conclusion, the rigid-viscoplastic constitutive laws for large strains in the *Eulerian approach* can be summarized in the following relationships:

$$\varphi_\alpha = N^\sigma_{\alpha ik}\sigma_{ik} - k_\alpha \leq 0 \tag{25}$$

$$D_{ij} = \sum_\alpha \dot{\lambda}_\alpha N^\sigma_{\alpha ij} \tag{26}$$

$$\dot{\lambda}_\alpha \geq 0, \quad \varphi_\alpha \dot{\lambda}_\alpha = 0 \tag{27}$$

$$D^0_{ij} = \sum_\alpha \ddot{\lambda}_\alpha \frac{\partial \varphi_\alpha}{\partial \sigma_{ij}} = \sum_\alpha \ddot{\lambda}_\alpha N^\sigma_{\alpha ij} \tag{28}$$

$$\varphi_\alpha \ddot{\lambda}_\alpha = 0 \text{ and } \ddot{\lambda}_\alpha \geq 0 \quad \text{if } \varphi_\alpha \leq 0 \text{ and } \dot{\lambda}_\alpha = 0 \ (V_R) \tag{29}$$

$$\varphi_\alpha \ddot{\lambda}_\alpha = 0 \text{ and } \ddot{\lambda}_\alpha \text{ free} \quad \text{if } \varphi_\alpha = 0 \text{ and } \dot{\lambda}_\alpha > 0 \ (V_P) \tag{30}$$

where eqns (29) and (30) define the rigid V_R and plastic V_P regions, respectively, and k_α is given by eqn (4).

2.2. *Lagrangian approach*

Let the assumed independent variables be the positions a_i occupied by a particle at time $t = 0$, i.e.

$$x_i = x_i(a_i) = a_i + u_i(a_i, t) \tag{31}$$

and let the stress and strain fields be described by Kirchhoff's and Green's tensors S_{ij}, E_{ij}, respectively. The yield domain, eqn (25), becomes

$$\varphi_\alpha = N^S_{\alpha hk} S_{hk} - k_\alpha \leqslant 0 \tag{32}$$

where

$$\begin{aligned} S_{hk} &= \frac{\rho_0}{\rho} \frac{\partial a_h}{\partial x_i} \frac{\partial a_k}{\partial x_j} \sigma_{ij} \\ \sigma_{ij} &= \frac{\rho}{\rho_0} \frac{\partial x_i}{\partial a_h} \frac{\partial x_j}{\partial a_k} S_{hk} \end{aligned} \tag{33}$$

ρ_0 and ρ being the initial and final density, and where, using eqn (25) once again

$$\begin{aligned} N^S_{\alpha hk} &= \frac{\rho}{\rho_0} \frac{\partial x_i}{\partial a_h} \frac{\partial x_j}{\partial a_k} N^\sigma_{\alpha ij} \\ N^\sigma_{\alpha ij} &= \frac{\rho_0}{\rho} \frac{\partial a_h}{\partial x_i} \frac{\partial a_k}{\partial x_j} N^S_{\alpha hk}. \end{aligned} \tag{34}$$

The flow rule, eqn (26), is transformed into

$$\dot{E}_{ij} = \sum_\alpha \dot{\lambda}_\alpha \frac{\rho_0}{\rho} N^S_{\alpha ij} \tag{35}$$

as can be easily shown starting from eqn (26) where $N^\sigma_{\alpha hk}$ is replaced by $N^S_{\alpha hk}$ using eqns (34) and taking into account the relation[4]

$$\dot{E}_{ij} = D_{hk} \frac{\partial x_h}{\partial a_i} \frac{\partial x_k}{\partial a_j} = \frac{\partial \dot{x}_k}{\partial a_i} \frac{\partial x_k}{\partial a_j} = \dot{F}_{ki} F_{kj}. \tag{36}$$

By a time derivative of eqn (35) we have finally

$$\ddot{E}_{ij} = \sum_\alpha \ddot{\lambda}_\alpha \frac{\rho_0}{\rho} N^S_{\alpha ij} + \sum_\alpha \dot{\lambda}_\alpha \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho_0}{\rho} N^S_{\alpha ij}\right). \tag{37}$$

Equation (37) shows that $\ddot{E}_{ij}$ is not normal to the yield surface. Taking into account eqns (A21), the last term of eqn (37) can be transformed as follows (denoting $\partial/\partial a_j$ with $/\bar{j}$):

$$\sum_\alpha \dot{\lambda}_\alpha \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho_0}{\rho} N^S_{\alpha ij}\right) = \sum_\alpha \dot{\lambda}_\alpha \hat{N}_{\alpha rs} \frac{\mathrm{d}}{\mathrm{d}t}(x_{h/\bar{i}} x_{k/\bar{j}} R_{hr} R_{ks}) \tag{37'}$$

and then using eqns (22), (10), (13), and (12) and remembering that $W_{ij} = \frac{1}{2}(\dot{u}_{i/j} - \dot{u}_{j/i})$

$$\begin{aligned} \sum_\alpha \dot{\lambda}_\alpha \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho_0}{\rho} N^S_{\alpha ij}\right) &= \sum_\alpha \dot{\lambda}_\alpha [N^\sigma_{\alpha hk}(\dot{u}_{h/\bar{i}} x_{k/\bar{j}} + x_{h/\bar{i}} \dot{u}_{k/\bar{j}}) \\ &\quad + N^\sigma_{\alpha mk} x_{h/\bar{i}} x_{k/\bar{j}} W_{hm} + N^\sigma_{\alpha hm} x_{h/\bar{i}} x_{k/\bar{j}} W_{km}] \\ &= \tfrac{1}{2}(\dot{u}_{k/\bar{i}} \dot{u}_{k/\bar{j}} + \dot{u}_{h/k} x_{h/\bar{i}} \dot{u}_{k/\bar{j}} + \dot{u}_{k/h} \dot{u}_{h/\bar{i}} x_{k/\bar{j}} \\ &\quad + \dot{u}_{k/m} x_{h/\bar{i}} x_{k/\bar{j}} \dot{u}_{h/m}). \end{aligned} \tag{38}$$

In conclusion, the rigid–viscoplastic constitutive laws for large strains in the *Lagrangian approach* can be summarized in the following relations:

$$\varphi_\alpha = N^S_{\alpha hk} S_{hk} - k_\alpha \leqslant 0 \tag{39}$$

$$\dot{E}_{ij} = \sum_\alpha \dot{\lambda}_\alpha \frac{\rho_0}{\rho} N^S_{\alpha ij} \tag{40}$$

$$\dot{\lambda}_\alpha \geqslant 0, \quad \varphi_\alpha \dot{\lambda}_\alpha = 0 \tag{41}$$

$$\ddot{E}_{ij} = \sum_\alpha \ddot{\lambda}_\alpha \frac{\rho_0}{\rho} N^S_{\alpha ij} + \sum_\alpha \dot{\lambda}_\alpha \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho_0}{\rho} N^S_{\alpha ij}\right) \tag{42}$$

$$\varphi_\alpha \ddot{\lambda}_\alpha = 0 \text{ and } \ddot{\lambda}_\alpha \geqslant 0 \quad \text{if } \varphi_\alpha \leqslant 0 \text{ and } \dot{\lambda}_\alpha = 0 \ (V_{R0}) \tag{43}$$

$$\varphi_\alpha \ddot{\lambda}_\alpha = 0 \text{ and } \ddot{\lambda}_\alpha \text{ free} \quad \text{if } \varphi_\alpha = 0 \text{ and } \dot{\lambda}_\alpha > 0 \ (V_{P0}) \tag{44}$$

where eqns (43) and (44) define the rigid V_{R0} and plastic V_{P0} regions, respectively, and k_α is given by eqn (4).

2.3. *Determination of plastic multiplier rates*

A problem ever encountered when solving a rigid–viscoplastic dynamic problem is find out whether at a given time t, a particular point with a known deformation gradient D_{ij} or $\dot{E}_{ij}$ lies in the plastic or rigid region. This can be accomplished simply by determining the plastic multiplier rate $\dot{\lambda}_\alpha$ corresponding to that particular D_{ij}, as $\dot{\lambda}_\alpha$ is positive in the plastic region only, whereas it vanishes in the rigid one.

With reference to the *Eulerian approach*, through the same proof used in Ref. [2] but replacing the flow rule by the condition

$$D_{ij} = \sum_\alpha N^\sigma_{\alpha ij} \dot{\lambda}_\alpha \tag{45}$$

it is possible to derive the following statement: the quadratic functional

$$\Omega(\dot{\lambda}^*_\alpha) = \frac{1}{2} \sum_{\alpha\beta} H_{\alpha\beta} \dot{\lambda}^*_\alpha \dot{\lambda}^*_\beta + \sum_\alpha r_\alpha \dot{\lambda}^*_\alpha \tag{46}$$

defined for all plastic multiplier rates $\dot{\lambda}^*_\alpha$ satisfying the conditions

$$\begin{aligned} D_{ij} &= \frac{1}{2}(\dot{u}_{i/j} + \dot{u}_{j/i}) = \sum_\alpha N^\sigma_{\alpha ij} \dot{\lambda}^*_\alpha \\ \dot{\lambda}^*_\alpha &\geqslant 0 \end{aligned} \tag{47}$$

attains an absolute minimum for the real value of the plastic multiplier rate $\dot{\lambda}_\alpha$.

If the hardening coefficients $H_{\alpha\beta}$ define a positive definite hardening matrix, the solution $\dot{\lambda}_\alpha$ will be unique.

The stress state σ_{ij} corresponding to the deformation tensor D_{ij} not always can be determined uniquely. In fact, the only limitations on the stresses σ_{ij} are set by conditions (27), i.e.

$$\begin{aligned} N^\sigma_{\alpha ij} \sigma_{ij} - k_\alpha &= 0 \quad \text{if} \quad \dot{\lambda}_\alpha > 0 \\ N^\sigma_{\alpha ij} \sigma_{ij} - k_\alpha &\leqslant 0 \quad \text{if} \quad \dot{\lambda}_\alpha = 0 \end{aligned} \tag{47'}$$

which may not be sufficient to determine σ_{ij} univocally.

Finally, using the *Lagrangian approach* the above statements still hold after replacing the normality condition (47) by the new one

$$\dot{E}_{ij} = \sum_{\alpha} \dot{\lambda}^*_{\alpha} \frac{\rho_0}{\rho} N^S_{\alpha ij}$$
$$\dot{\lambda}^*_{\alpha} \geqslant 0. \tag{47''}$$

3. FORMULATION OF THE DYNAMIC-LOADING PROBLEM

3.1. *Eulerian approach*

Let us assume that, at a given time t, the current configuration u_i of a body having volume V and surface S, the density ρ, the field of velocities $\dot{u}_i$, the surface tractions T_i on a part S_T of S, the surface accelerations $\ddot{U}_i$ over the remaining part $S_u = S - S_T$ of the surface, and the body forces $X_i = \rho F_i$ are known.

The dynamic-loading problem consists in determining the stress field σ_{ij} and the acceleration field $\ddot{u}_i$ for the same instant t. From the solution of this problem, when dealing with time-dependent known quantities $X_i(t)$, $T_i(t)$, $u_i(t)$, $\dot{u}_i(t)$ and $\ddot{U}_i(t)$, the whole dynamic history, i.e. the unknown functions $\ddot{u}_i(t)$, $\sigma_{ij}(t)$ can be obtained simply by an incremental process over time t.

All the conditions governing the dynamic-loading problem at a generic instant t are given below.

(a) Equilibrium equations

$$\begin{aligned} \sigma_{ij/j} + \rho F_i &= \rho \ddot{u}_i \quad \text{in } V \\ \sigma_{ij} n_j &= T_i \quad \text{on } S_T \end{aligned} \tag{48}$$

where σ_{ij} are the components of the Cauchy stress tensor.

(b) Compatibility equations

$$\begin{aligned} e_{ij} &= \tfrac{1}{2}(u_{i/j} + u_{j/i} - u_{k/i} u_{k/j}) \\ \dot{e}_{ij} &= \tfrac{1}{2}(\dot{u}_{i/j} + \dot{u}_{j/i}) - (e_{ik}\dot{u}_{k/j} + \dot{u}_{k/i} e_{kj}) \\ \ddot{e}_{ij} &= \dot{D}_{ij} - (D_{ik}\dot{u}_{k/j} - e_{ih}\dot{u}_{h/k}\dot{u}_{k/j} - \dot{u}_{h/i} e_{hk}\dot{u}_{k/j} \\ &\quad + e_{ik}(\ddot{u}_{k/j} - \dot{u}_{h/j}\dot{u}_{k/h}) + e_{kj}(\ddot{u}_{k/i} - \dot{u}_{h/i}\dot{u}_{k/h}) \\ &\quad + \dot{u}_{k/i} D_{kj} - \dot{u}_{k/i} e_{kh}\dot{u}_{h/j} - \dot{u}_{k/i}\dot{u}_{h/k} e_{hj}) \quad \text{in } V \\ \ddot{u}_j &= \ddot{U}_j \quad \text{on } S_u \end{aligned} \tag{49}$$

where e_{ij} are the components of the Almansi strain tensor; besides

$$\dot{D}_{ij} = \tfrac{1}{2}(\ddot{u}_{i/j} + \ddot{u}_{j/i}) - \tfrac{1}{2}(\dot{u}_{k/j}\dot{u}_{i/k} + \dot{u}_{k/i}\dot{u}_{j/k}) \tag{50}$$

and the objective derivatives D_{ij} of e_{ij} and D^0_{ij} of $\dot{e}_{ij}$ are

$$\begin{aligned} D_{ij} &= \tfrac{1}{2}(\dot{u}_{i/j} + \dot{u}_{j/i}) \\ D^0_{ij} &= \tfrac{1}{2}(\ddot{u}_{i/j} + \ddot{u}_{j/i}) - \tfrac{1}{2}(\dot{u}_{k/j}\dot{u}_{i/k} + \dot{u}_{k/i}\dot{u}_{j/k} + \dot{u}_{i/k}\dot{u}_{j/k} - \dot{u}_{k/i}\dot{u}_{k/j}). \end{aligned} \tag{51}$$

(c) Constitutive laws, given by eqns (25)–(30). The velocity field $\dot{\lambda}_{\alpha}$ appearing in the above equations can be obtained as indicated in Section 2.3 form the known field $\dot{u}_i$ (which define D_{ij} according eqn $(51)_1$).

3.2. *Lagrangian approach*

All the equations governing the dynamic loading problem at a generic instant t are given below.

(a) Equilibrium equations

$$\frac{\partial}{\partial a_j}\left(S_{jk}\frac{\partial x_i}{\partial a_k}\right)+\rho_0 F_{0i} = \rho_0 \ddot{u}_i \quad \text{in } V_0$$
$$S_{jk}\frac{\partial x_i}{\partial a_k}n_{0j} = T_{0i} \quad \text{on } S_{0T} \tag{52}$$

where S_{ij} are the components of the Kirchhoff symmetric stress tensor.

(b) Compatibility equations (denoting $\partial/\partial a_j$ with $/\bar{j}$)

$$E_{ij} = \tfrac{1}{2}(u_{i/\bar{j}}+u_{j/\bar{i}}+u_{k/\bar{i}}u_{k/\bar{j}})$$
$$\dot{E}_{ij} = \tfrac{1}{2}(\dot{u}_{i/\bar{j}}+\dot{u}_{j/\bar{i}}+\dot{u}_{k/\bar{i}}u_{k/\bar{j}}+u_{k/\bar{i}}\dot{u}_{k/\bar{j}})$$
$$\ddot{E}_{ij} = \tfrac{1}{2}(\ddot{u}_{i/\bar{j}}+\ddot{u}_{j/\bar{i}}+\ddot{u}_{k/\bar{i}}u_{k/\bar{j}}+u_{k/\bar{i}}\ddot{u}_{k/\bar{j}})+\dot{u}_{k/\bar{i}}\dot{u}_{k/\bar{j}} \quad \text{in } V_0$$
$$\ddot{u}_i = \ddot{U}_{i0} \quad \text{on } S_{0u} \tag{53}$$

where E_{ij} are the components of the Green strain tensor and $\dot{E}_{ij}$ and $\ddot{E}_{ij}$ are objective by definition.

(c) Constitutive laws, given by eqns (39)–(44). The velocity field $\dot{\lambda}_\alpha$ appearing in the above equations can be obtained as indicated in Section 2.3 from the known fields u_i and $\dot{u}_i$ (which define $\dot{E}_{ij}$ according eqn $(53)_2$).

4. MINIMUM PRINCIPLES

4.1. *Eulerian approach*

Theorem I. *The functional*

$$\Phi(\sigma^*_{ij},\ddot{u}^*_i) = \frac{1}{2}\int_V \rho\ddot{u}^*_j\ddot{u}^*_j\,\mathrm{d}V+\frac{1}{2}\int_V \sigma^*_{ij}[\dot{u}_{i/k}\dot{u}_{k/j}+\dot{u}_{k/i}\dot{u}_{j/k}+\dot{u}_{i/k}\dot{u}_{j/k}-\dot{u}_{k/i}\dot{u}_{k/j}]\,\mathrm{d}V - \int_{S_u}\sigma^*_{ij}n_i\ddot{U}_j\,\mathrm{d}S_u \tag{54}$$

defined for all stresses σ^*_{ij} and accelerations $\ddot{u}^*_j$ satisfying the conditions

$$\sigma^*_{ij/i}+\rho F_j = \rho\ddot{u}^*_j \quad \text{in } V$$
$$\sigma^*_{ij}n_i = T_j \quad \text{on } S_T$$
$$N^\sigma_{\alpha ik}\sigma^*_{ik}-k_\alpha = 0 \quad \text{in } V_P$$
$$N^\sigma_{\alpha ik}\sigma^*_{ik}-k_\alpha \leqslant 0 \quad \text{in } V_R \tag{55}$$

attains an absolute minimum for the actual stress σ_{ij} and accelerations $\ddot{u}_j$ fields, V_P and V_R being the plastic and rigid regions defined by conditions (29) *and* (30), *respectively.*

Proof. In order to prove the theorem, it suffices to show that the difference

$$\Delta\Phi = \Phi(\sigma^*_{ij},\ddot{u}^*_j)-\Phi(\sigma_{ij},\ddot{u}_j) \tag{56}$$

is always nonnegative for any arbitrary σ^*_{ij}, $\ddot{u}^*_j$ satisfying conditions (55). Assuming

$$\Delta\sigma_{ij} = \sigma^*_{ij} - \sigma_{ij}$$
$$\Delta\ddot{u}_j = \ddot{u}^*_j - \ddot{u}_j \tag{57}$$

the difference, eqn (56), becomes

$$\Delta\Phi = \frac{1}{2}\int_V \rho\Delta\ddot{u}_j\Delta\ddot{u}_j\,\mathrm{d}V + \int_V \rho\Delta\ddot{u}_j\ddot{u}_j\,\mathrm{d}V + \frac{1}{2}\int_V \Delta\sigma_{ij}[\dot{u}_{i|k}\dot{u}_{k|j} + \dot{u}_{k|i}\dot{u}_{j|k} + \dot{u}_{i|k}\dot{u}_{j|k} - \dot{u}_{k|i}\dot{u}_{k|j}]\,\mathrm{d}V$$
$$-\int_{S_u} \Delta\sigma_{ij}n_i\ddot{U}_j\,\mathrm{d}S_u. \tag{58}$$

Using Gauss's theorem and eqns $(55)_1$ and $(55)_2$

$$\int_V \rho\Delta\ddot{u}_j\ddot{u}_j\,\mathrm{d}V = \int_V \Delta\sigma_{ij|i}\ddot{u}_j\,\mathrm{d}V = -\int_V \Delta\sigma_{ij}\ddot{u}_{j|i}\,\mathrm{d}V + \int_{S_u} \Delta\sigma_{ij}n_i\ddot{U}_j\,\mathrm{d}S_u + \int_{S_T} \Delta\sigma_{ij}n_i\ddot{U}_j\,\mathrm{d}S_T \tag{59}$$

where the last term of the second member vanishes because $\Delta\sigma_{ij} = 0$ on S_T. Then the difference, eqn (56), becomes

$$\Delta\Phi = \frac{1}{2}\int_V \rho\Delta\ddot{u}_j\Delta\ddot{u}_j\,\mathrm{d}V - \int_V \Delta\sigma_{ij}\ddot{u}_{j|i}\,\mathrm{d}V + \frac{1}{2}\int_V \Delta\sigma_{ij}[\dot{u}_{i|k}\dot{u}_{k|j} + \dot{u}_{k|i}\dot{u}_{j|k} + \dot{u}_{i|k}\dot{u}_{j|k} - \dot{u}_{k|i}\dot{u}_{k|j}]\,\mathrm{d}V \tag{60}$$

where the second integral, owing to the symmetry of the Cauchy stress tensor, may be written in the form

$$-\int_V \Delta\sigma_{ij}\ddot{u}_{j|i}\,\mathrm{d}V = -\frac{1}{2}\int_V \Delta\sigma_{ij}(\ddot{u}_{i|j} + \ddot{u}_{j|i})\,\mathrm{d}V \tag{61}$$

and the difference $\Delta\Phi$, by virtue of eqn (18), becomes

$$\Delta\Phi = \frac{1}{2}\int_V \rho\Delta\ddot{u}_j\Delta\ddot{u}_j\,\mathrm{d}V - \int_V \Delta\sigma_{ij}D^0_{ij}\,\mathrm{d}V. \tag{62}$$

The first term on the right-hand side is always positive for any non-vanishing $\Delta\ddot{u}_j$ and equal to zero if and only if

$$\ddot{u}^*_i = \ddot{u}_i \quad \text{in } V. \tag{62'}$$

Using eqn (28) the second term becomes

$$-\int_V \Delta\sigma_{ij}D^0_{ij}\,\mathrm{d}V = -\sum_\alpha \int_V \Delta\sigma_{ij}N^\sigma_{\alpha ij}\dot{\lambda}_\alpha\,\mathrm{d}V. \tag{63}$$

In region V_P (where $\varphi_\alpha = 0$ and $\dot{\lambda}_\alpha > 0$), from eqns $(55)_3$ and (30) it can be stated that

$$-\sum_{\alpha}\int_{V}\Delta\sigma_{ij}N^{\sigma}_{\alpha ij}\ddot{\lambda}_{\alpha}\,\mathrm{d}V=\sum_{\alpha}\int_{V}(N^{\sigma}_{\alpha ij}\sigma_{ij}-k_{\alpha})\ddot{\lambda}_{\alpha}\,\mathrm{d}V=\sum_{\alpha}\int_{V}\varphi_{\alpha}\ddot{\lambda}_{\alpha}\,\mathrm{d}V=0. \quad (64)$$

In the remaining region V_R (where $\dot{\lambda}_\alpha = 0$), from the condition

$$\varphi_\alpha\ddot{\lambda}_\alpha = (N^{\sigma}_{\alpha ij}\sigma_{ij}-k_\alpha)\ddot{\lambda}_\alpha = 0 \quad (65)$$

we can write eqn (63) in the form

$$-\sum_{\alpha}\int_{V}\Delta\sigma_{ij}N^{\sigma}_{\alpha ij}\ddot{\lambda}_{\alpha}\,\mathrm{d}V = -\sum_{\alpha}\int_{V}(N^{\sigma}_{\alpha ij}\sigma^{*}_{ij}-k_{\alpha})\ddot{\lambda}_{\alpha}\,\mathrm{d}V \quad (66)$$

which will never be negative as a result of condition $(55)_4$ and of the nonnegativity of $\ddot{\lambda}_\alpha$ in V_R stated in eqn (29). It will be equal to zero if and only if

$$N^{\sigma}_{\alpha ij}\sigma^{*}_{ij}-k_\alpha = 0 \quad \text{where } \ddot{\lambda}_\alpha > 0. \quad (66')$$

This proves that

$$\Phi(\sigma^{*}_{ij},\ddot{u}^{*}_{j}) \geq \Phi(\sigma_{ij},\ddot{u}_{j}) \quad (67)$$

for any stress σ^{*}_{ij} and acceleration $\ddot{u}^{*}_{j}$ fields satisfying conditions (55), the equality sign holding if and only if the accelerations $\ddot{u}^{*}_{j}$ satisfy eqn (62′) and stresses σ^{*}_{ij} satisfy eqn (66′).

This proves the theorem. It follows from this proof that the actual acceleration field $\ddot{u}_i$ is univocally defined; on the contrary, the actual stress field σ_{ij} may not be determined univocally because the only limitations that we must respect are

$$\begin{aligned} &N^{\sigma}_{\alpha ij}\sigma^{*}_{ij}-k_\alpha = 0 \quad \text{in } V_P\\ &N^{\sigma}_{\alpha ij}\sigma^{*}_{ij}-k_\alpha = 0 \quad \text{if } \ddot{\lambda}_\alpha > 0 \quad \text{in } V_R\\ &N^{\sigma}_{\alpha ij}\sigma^{*}_{ij}-k_\alpha \leq 0 \quad \text{if } \ddot{\lambda}_\alpha = 0 \quad \text{in } V_R \end{aligned} \quad (67')$$

which may not always be sufficient to determine σ^{*}_{ij} univocally.

Theorem II. *The functional*

$$\Psi(\ddot{u}^{*}_{j},\ddot{\lambda}^{*}_{\alpha}) = \frac{1}{2}\int_{V}\rho\ddot{u}^{*}_{j}\ddot{u}^{*}_{j}\,\mathrm{d}V-\int_{V}\rho F_j\ddot{u}^{*}_{j}\,\mathrm{d}V-\int_{S_T}T_j\ddot{u}^{*}_{j}\,\mathrm{d}S_T+\sum_{\alpha}\int_{V}k_\alpha\ddot{\lambda}^{*}_{\alpha}\,\mathrm{d}V \quad (68)$$

defined for all accelerations $\ddot{u}^{*}_{j}$ *and plastic multiplier accelerations* $\ddot{\lambda}^{*}_{\alpha}$ *satisfying the conditions*

$$\begin{aligned} &\ddot{u}^{*}_{j} = \ddot{U}_j \quad \text{on } S_u\\ &\sum_{\alpha}N^{\sigma}_{\alpha ij}\ddot{\lambda}^{*}_{\alpha} = \frac{1}{2}(\ddot{u}^{*}_{i,j}+\ddot{u}^{*}_{j,i})-\frac{1}{2}(\dot{u}_{k,i}\dot{u}_{j,k}+\dot{u}_{k,j}\dot{u}_{i,k}+\dot{u}_{i,k}\dot{u}_{j,k}-\dot{u}_{k,i}\dot{u}_{k,j}) = D^{0*}_{ij} \quad \text{in } V\\ &\ddot{\lambda}^{*}_{\alpha} \geq 0 \quad \text{in } V_R \end{aligned} \quad (69)$$

attains an absolute minimum for the actual accelerations $\ddot{u}_j$ *and plastic multiplier accelerations* $\ddot{\lambda}_\alpha$, V_R *being the rigid region as defined by eqn* (29).

Proof. The following proves that the difference

$$\Delta\Psi = \Psi(\ddot{u}^{*}_{j},\ddot{\lambda}^{*}_{\alpha})-\Psi(\ddot{u}_{j},\ddot{\lambda}_{\alpha}) \quad (70)$$

is always nonnegative for any arbitrary $\ddot{u}^{*}_{j}$, $\ddot{\lambda}^{*}_{\alpha}$ satisfying conditions (69). Assuming

$$\begin{aligned}\Delta\ddot{u}_j &= \ddot{u}_j^* - \ddot{u}_j \\ \Delta\dot{\lambda}_\alpha &= \dot{\lambda}_\alpha^* - \dot{\lambda}_\alpha\end{aligned} \tag{71}$$

the above difference becomes

$$\Delta\Psi = \frac{1}{2}\int_V \rho\Delta\ddot{u}_j\Delta\ddot{u}_j\,\mathrm{d}V + \int_V \rho\ddot{u}_j\Delta\ddot{u}_j\,\mathrm{d}V - \int_V \rho F_j\Delta\ddot{u}_j\,\mathrm{d}V - \int_{S_T} T_j\Delta\ddot{u}_j\,\mathrm{d}S_T + \sum_\alpha \int_V k_\alpha\Delta\dot{\lambda}_\alpha\,\mathrm{d}V. \tag{72}$$

Using the equilibrium equations, eqns (48), and Gauss's theorem we obtain

$$\begin{aligned}\int_V \rho\ddot{u}_j\Delta\ddot{u}_j\,\mathrm{d}V &= \int_V \sigma_{ij/i}\Delta\ddot{u}_j\,\mathrm{d}V + \int_V \rho F_j\Delta\ddot{u}_j\,\mathrm{d}V \\ &= \int_{S_T} T_j\Delta\ddot{u}_j\,\mathrm{d}S_T - \int_V \sigma_{ij}\Delta\ddot{u}_{j/i}\,\mathrm{d}V + \int_V \rho F_j\Delta\ddot{u}_j\,\mathrm{d}V\end{aligned} \tag{73}$$

and consequently the difference $\Delta\Psi$ reduces to

$$\Delta\Psi = \frac{1}{2}\int_V \rho\Delta\ddot{u}_j\Delta\ddot{u}_j\,\mathrm{d}V - \int_V \sigma_{ij}\Delta\ddot{u}_{j/i}\,\mathrm{d}V + \sum_\alpha \int_V k_\alpha\Delta\dot{\lambda}_\alpha\,\mathrm{d}V. \tag{74}$$

The symmetry of the Cauchy stress tensor and the use of eqn $(51)_2$, enable us to write the second integral of eqn (74) in the form

$$-\int_V \sigma_{ij}\Delta\ddot{u}_{j/i}\,\mathrm{d}V = -\frac{1}{2}\int_V \sigma_{ij}(\Delta\ddot{u}_{i/j} + \Delta\ddot{u}_{j/i})\,\mathrm{d}V = -\int_V \sigma_{ij}\Delta D_{ij}^0\,\mathrm{d}V. \tag{75}$$

Then

$$\Delta\Psi = \frac{1}{2}\int_V \rho\Delta\ddot{u}_j\Delta\ddot{u}_j\,\mathrm{d}V - \int_V \sigma_{ij}\Delta D_{ij}^0\,\mathrm{d}V + \sum_\alpha \int_V k_\alpha\Delta\dot{\lambda}_\alpha\,\mathrm{d}V \tag{76}$$

which, using condition $(69)_2$, is transformed into

$$\Delta\Psi = \frac{1}{2}\int_V \rho\Delta\ddot{u}_j\Delta\ddot{u}_j\,\mathrm{d}V - \sum_\alpha \int_V (N_{\alpha ij}^\sigma\sigma_{ij} - k_\alpha)\Delta\dot{\lambda}_\alpha\,\mathrm{d}V. \tag{77}$$

The first integral is always positive and equal to zero if and only if

$$\ddot{u}_j^* = \ddot{u}_j \quad \text{in } V. \tag{77'}$$

The second integral, in the region V_P vanishes, because $\varphi_\alpha = 0$; in the remaining region V_R, by virtue of the condition

$$\varphi_\alpha\dot{\lambda}_\alpha = (N_{\alpha ij}^\sigma\sigma_{ij} - k_\alpha)\dot{\lambda}_\alpha = 0 \tag{78}$$

the term reduces

$$-\sum_\alpha \int_V (N^\sigma_{\alpha ij}\sigma_{ij}-k_\alpha)\Delta\ddot{\lambda}_\alpha \, \mathrm{d}V = -\sum_\alpha \int_{V_R} (N^\sigma_{\alpha ij}\sigma_{ij}-k_\alpha)\ddot{\lambda}^*_\alpha \, \mathrm{d}V \tag{79}$$

which is always nonnegative for every non-vanishing $\ddot{\lambda}^*_\alpha$ in consequence of condition $(69)_3$ and because $\varphi_\alpha \leqslant 0$, and is zero if and only if

$$\ddot{\lambda}^*_\alpha = 0 \quad \text{where} \quad N^\sigma_{\alpha ij}\sigma_{ij}-k_\alpha < 0 \quad \text{in } V_R. \tag{79'}$$

This proves that

$$\Psi(\ddot{u}^*_i, \ddot{\lambda}^*_\alpha) \geqslant \Psi(\ddot{u}_i, \ddot{\lambda}_\alpha) \tag{80}$$

for any acceleration field $\ddot{u}^*_i$ and plastic multiplier accelerations $\ddot{\lambda}^*_\alpha$ satisfying conditions (69), the equality sign holds only inasmuch as eqns (77′) and (79′) are satisfied.

This proves the theorem. Again, it follows from this proof that the actual acceleration field $\ddot{u}_j$ is univocally defined; on the contrary, the actual plastic multiplier accelerations $\ddot{\lambda}_\alpha$ may not be determined univocally (even for non-vanishing hardening coefficients $H_{\alpha\beta}$), because the only limitations that $\ddot{\lambda}^*_\alpha$ must satisfy are

$$\begin{aligned} &\sum_\alpha N^\sigma_{\alpha ij}\ddot{\lambda}^*_\alpha = D^{0*}_{ij} \\ &\ddot{\lambda}^*_\alpha \geqslant 0 \text{ if } N^\sigma_{\alpha ij}\sigma_{ij}-k_\alpha = 0 \quad \text{in } V_R \\ &\ddot{\lambda}^*_\alpha = 0 \text{ if } N^\sigma_{\alpha ij}\sigma_{ij}-k_\alpha < 0 \quad \text{in } V_R \end{aligned} \tag{80'}$$

which may not be sufficient to determine $\ddot{\lambda}^*_\alpha$ univocally.

4.2. *Lagrangian approach*

A simple way to derive *a theorem corresponding to Theorem* I but involving quantities relevant to the Lagrangian approach only, is to adopt a similar proof as Theorem I starting from the following statement.

The functional

$$\Phi^L(S^*_{ij}, \ddot{u}^*_j) = \frac{1}{2}\int_{V_0} \rho_0\ddot{u}^*_j\ddot{u}^*_j \, \mathrm{d}V_0 + \frac{1}{2}\int_{V_0} S^*_{ij}(x_{h/i}\dot{u}_{r/j}\dot{u}_{h/r} + x_{h/i}x_{k/j}\dot{u}_{h/r}\dot{u}_{k/r} + \dot{u}_{r/i}x_{k/j}\dot{u}_{k/r} - \dot{u}_{r/i}\dot{u}_{r/j}) \, \mathrm{d}V_0 - \int_{S_{u0}} x_{k/j}S^*_{ij}n_{0i}\ddot{U}_k \, \mathrm{d}S_{u0} \tag{81}$$

defined for all stress S^*_{ij} *and acceleration* $\ddot{u}^*_j$ *fields satisfying the conditions*

$$\begin{aligned} &\frac{\partial}{\partial a_j}\left(S^*_{jk}\frac{\partial x_i}{\partial a_k}\right) + \rho_0 F_{0i} = \rho_0\ddot{u}^*_i \quad \text{in } V_0 \\ &S^*_{jk}\frac{\partial x_i}{\partial a_k}n_{0j} = T_{0j} \quad \text{on } S_{T0} \\ &N^S_{\alpha ik}S^*_{ik} - k_\alpha = 0 \quad \text{in } V_{P0} \\ &N^S_{\alpha ik}S^*_{ik} - k_\alpha \leqslant 0 \quad \text{in } V_{R0} \end{aligned} \tag{82}$$

attains an absolute minimum for the actual stresses S_{ij} *and accelerations* $\ddot{u}_i$, V_{P0} *and* V_{R0} *being the plastic and rigid regions referred to the initial configuration.*

Again, the proof consists in showing that the difference

$$\Delta\Phi^L = \Phi^L(S^*_{ij}, \ddot{u}^*_j) - \Phi^L(S_{ij}, \ddot{u}_j) \tag{83}$$

is always nonnegative for any arbitrary S^*_{ij}, $\ddot{u}^*_j$ satisfying conditions (82).

Assuming

$$\begin{aligned} \Delta S_{ij} &= S^*_{ij} - S_{ij} \\ \Delta\ddot{u}_j &= \ddot{u}^*_j - \ddot{u}_j \\ A_{ij} &= x_{h,\bar{i}}\dot{u}_{r,\bar{j}}\dot{u}_{h,r} + x_{h,\bar{i}}x_{k,\bar{j}}\dot{u}_{h,r}\dot{u}_{k,r} + \dot{u}_{r,\bar{i}}x_{k,\bar{j}}\dot{u}_{k,r} + \dot{u}_{r,\bar{i}}\dot{u}_{r,\bar{j}} \\ B_{ij} &= A_{ij} - 2\dot{u}_{r,\bar{i}}\dot{u}_{r,\bar{j}} \end{aligned} \tag{84}$$

we have

$$\Delta\Phi^L = \frac{1}{2}\int_{V_0} \rho_0 \Delta\ddot{u}_i \Delta\ddot{u}_i \, \mathrm{d}V_0 + \int_{V_0} \rho_0 \Delta\ddot{u}_i \ddot{u}_i \, \mathrm{d}V_0 + \frac{1}{2}\int_{V_0} \Delta S_{ij} B_{ij} \, \mathrm{d}V_0 - \int_{S_{u0}} \frac{\partial x_k}{\partial a_j} \Delta S_{ij} n_{0i} \ddot{U}_k \, \mathrm{d}S_{u0}. \tag{85}$$

Using (equilibrium) eqns $(82)_1$ and $(82)_2$ and Gauss's theorem, we obtain

$$\begin{aligned} \int_{V_0} \rho_0 \Delta\ddot{u}_i \ddot{u}_i \, \mathrm{d}V_0 &= \int_{V_0} \left(\Delta S_{jk} \frac{\partial x_i}{\partial a_k}\right)_{,\bar{j}} \ddot{u}_i \, \mathrm{d}V_0 \\ &= -\int_{V_0} \Delta S_{jk} \frac{\partial x_i}{\partial a_k} \ddot{u}_{i,\bar{j}} \, \mathrm{d}V_0 + \int_{S_{u0}} \Delta S_{jk} \frac{\partial x_i}{\partial a_k} n_{0j} \ddot{U}_i \, \mathrm{d}S_{u0} \\ &\quad + \int_{S_{f0}} \Delta S_{jk} \frac{\partial x_i}{\partial a_k} n_{0i} \ddot{U}_j \, \mathrm{d}S_{f0} \end{aligned} \tag{86}$$

where the last term on the right-hand side vanishes because $\Delta S_{jk} = 0$ on S_{f0}. Then the difference, eqn (85), becomes

$$\Delta\Phi^L = \frac{1}{2}\int_{V_0} \rho_0 \Delta\ddot{u}_i \Delta\ddot{u}_i \, \mathrm{d}V_0 - \int_{V_0} \Delta S_{jk}\left(\frac{\partial x_i}{\partial a_k}\ddot{u}_{i,\bar{j}}\right) \mathrm{d}V_0 + \frac{1}{2}\int_{V_0} \Delta S_{ij} B_{ij} \, \mathrm{d}V_0$$

where on account of the symmetry of the Kirchhoff stress tensor, and using eqns (A3), the second integral may be written as

$$\begin{aligned} -\int_{V_0} \Delta S_{jk}\left(\frac{\partial x_i}{\partial a_k}\ddot{u}_{i,\bar{j}}\right) \mathrm{d}V_0 &= -\frac{1}{2}\int_{V_0} \Delta S_{jk}\left[\frac{\partial x_i}{\partial a_k}\ddot{u}_{i,\bar{j}} + \frac{\partial x_i}{\partial a_j}\ddot{u}_{i,\bar{k}}\right] \mathrm{d}V_0 \\ &= -\frac{1}{2}\int_{V_0} \Delta S_{jk}[\ddot{u}_{k,\bar{j}} + \ddot{u}_{i,\bar{j}}u_{i,\bar{k}} + \ddot{u}_{j,\bar{k}} + \ddot{u}_{i,\bar{k}}u_{i,\bar{j}}] \, \mathrm{d}V_0 \end{aligned} \tag{87}$$

and the difference $\Delta\Phi^L$, by virtue of eqns (53), becomes

$$\Delta\Phi^L = \frac{1}{2}\int_{V_0} \rho_0 \Delta\ddot{u}_i \Delta\ddot{u}_i \, \mathrm{d}V_0 - \int_{V_0} \Delta S_{ij} \ddot{E}_{ij} \, \mathrm{d}V_0 + \frac{1}{2}\int_{V_0} \Delta S_{ij} A_{ij} \, \mathrm{d}V_0. \tag{88}$$

In eqn (88) the first integral of the second member is always positive for every non-vanishing $\Delta\ddot{u}_i$ and equal to zero if and only if

$$\ddot{u}_i^* = \ddot{u}_i \quad \text{in } V_0. \tag{89}$$

In consideration of eqn (42) the second term on the right-hand side of eqn (88) becomes

$$-\int_{V_0} \Delta S_{ij}\ddot{E}_{ij}\, dV_0 = -\sum_\alpha \int_{V_0} \Delta S_{ij}\ddot{\lambda}_\alpha \frac{\rho_0}{\rho} N^S_{\alpha ij}\, dV_0 - \sum_\alpha \int_{V_0} \Delta S_{ij}\dot{\lambda}_\alpha \frac{d}{dt}\left(\frac{\rho_0}{\rho} N^S_{\alpha ij}\right) dV_0 \tag{90}$$

where, using eqn (38)

$$-\sum_\alpha \int_{V_0} \Delta S_{ij}\dot{\lambda}_\alpha \frac{d}{dt}\left(\frac{\rho_0}{\rho} N^S_{\alpha ij}\right) dV_0 = -\frac{1}{2}\int_{V_0} \Delta S_{ij} A_{ij}\, dV_0. \tag{90'}$$

Then the difference $\Delta\Phi^L$ of eqn (88) becomes

$$\Delta\Phi^L = \frac{1}{2}\int_{V_0} \rho_0 \Delta\ddot{u}_i \Delta\ddot{u}_i\, dV_0 - \sum_\alpha \int_{V_0} \Delta S_{ij}\ddot{\lambda}_\alpha \frac{\rho_0}{\rho} N^S_{\alpha ij}\, dV_0. \tag{91}$$

In *the plastic region* (where $\varphi_\alpha = 0$, $\dot{\lambda}_\alpha \geq 0$, $\varphi_\alpha \dot{\lambda}_\alpha = 0$), from condition $(82)_2$ and (30), it can be stated that

$$-\sum_\alpha \int_{V_0} \Delta S_{ij}\ddot{\lambda}_\alpha \frac{\rho_0}{\rho} N^S_{\alpha ij}\, dV_0 = \sum_\alpha \int_{V_0} (N^S_{\alpha ij} S_{ij} - k_\alpha)\ddot{\lambda}_\alpha \frac{\rho_0}{\rho}\, dV_0 = \sum_\alpha \int_{V_0} \varphi_\alpha \ddot{\lambda}_\alpha \frac{\rho_0}{\rho}\, dV_0 = 0.$$

In the remaining region (where $\dot{\lambda}_\alpha = 0$), from the condition

$$\varphi_\alpha \ddot{\lambda}_\alpha = (N^S_{\alpha ij} S_{ij} - k_\alpha)\ddot{\lambda}_\alpha = 0$$

we can rewrite the last term of eqn (91) in the form

$$-\sum_\alpha \int_{V_0} \Delta S_{ij}\ddot{\lambda}_\alpha \frac{\rho_0}{\rho} N^S_{\alpha ij}\, dV_0 = -\sum_\alpha \int_{V_0} (N^S_{\alpha ij} S^*_{ij} - k_\alpha)\ddot{\lambda}_\alpha \frac{\rho_0}{\rho}\, dV_0 \tag{92}$$

which will never be negative on account of condition $(82)_4$ and of the nonnegativity of $\ddot{\lambda}_\alpha$ in V_{R0} stated in eqn (43).

This will be equal to zero if and only if

$$N^S_{\alpha ij} S^*_{ij} - k_\alpha = 0 \quad \text{where } \ddot{\lambda}_\alpha > 0. \tag{93}$$

This proves that

$$\Phi^L(S^*_{ij}, \ddot{u}^*_i) \geq \Phi^L(S_{ij}, \ddot{u}_i)$$

for any stress S^*_{ij} and acceleration $\ddot{u}^*_i$ satisfying conditions (82), the equality sign holding if, and only if, the accelerations $\ddot{u}^*_i$ satisfy eqn (89) and the stresses S^*_{ij} satisfy eqn (93). This proves the theorem.

It follows from the above proof that the actual acceleration field $\ddot{u}_i$ is univocally defined; on the contrary, the actual stress field S_{ij} may not be univocally determined because the only limitations that we must respect are

$$
\begin{aligned}
&N^S_{\alpha ij}S^*_{ij}-k_\alpha = 0 \quad \text{in } V_{P0}\\
&N^S_{\alpha ij}S^*_{ij}-k_\alpha = 0 \quad \text{if } \ddot{\lambda}_\alpha > 0 \quad \text{in } V_{R0}\\
&N^S_{\alpha ij}S^*_{ij}-k_\alpha \leqslant 0 \quad \text{if } \ddot{\lambda}_\alpha = 0 \quad \text{in } V_{R0}
\end{aligned}
\tag{94}
$$

which may not always be sufficient to determine S_{ij} univocally.

Finally it is easy to show that the statement of the present theorem could have been obtained starting from the corresponding Eulerian formulation (54), (55) and using the formal relationships (see Appendix) between the variables of the Euleriàn and Lagrangian approach.

A theorem corresponding to Theorem II but involving quantities relevant to the Lagrangian approach can be derived by a similar proof, starting from the following statement.

The functional

$$
\Psi^{\mathrm{L}}(\ddot{u}_i^*,\ddot{\lambda}_\alpha) = \frac{1}{2}\int_{V_0}\rho_0\ddot{u}_i^*\ddot{u}_i^*\,\mathrm{d}V_0-\int_{V_0}\rho_0F_{0i}\ddot{u}_i^*\,\mathrm{d}V_0-\int_{S_{T0}}T_{0i}\ddot{u}_i^*\,\mathrm{d}S_{T0}+\sum_\alpha\int_{V_0}k_\alpha\ddot{\lambda}_\alpha^*\frac{\rho_0}{\rho}\,\mathrm{d}V_0
\tag{95}
$$

defined for all plastic multiplier accelerations $\ddot{\lambda}_\alpha^*$ and accelerations $\ddot{u}_i^*$ satisfying conditions

$$
\begin{aligned}
&\ddot{u}_i^* = \ddot{U}_i \quad \text{on } S_{u0}\\
&\ddot{E}_{ij}^* = \sum_\alpha\ddot{\lambda}_\alpha\left(\frac{\rho_0}{\rho}N^S_{\alpha ij}\right)+\sum_\alpha\dot{\lambda}_\alpha\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho_0}{\rho}N^S_{\alpha ij}\right) \quad \text{in } V_0\\
&\ddot{\lambda}_\alpha^* \geqslant 0 \quad \text{in } V_{R0}
\end{aligned}
\tag{96}
$$

attains an absolute minimum for the actual accelerations $\ddot{u}_i$ and plastic multiplier accelerations $\ddot{\lambda}_\alpha$, V_{R0} being the rigid region referred to the initial configuration.

Again, the point to be proved is that the difference

$$
\Delta\Psi^{\mathrm{L}} = \Psi^{\mathrm{L}}(\ddot{u}_i^*,\ddot{\lambda}_\alpha^*)-\Psi^{\mathrm{L}}(\ddot{u}_i,\ddot{\lambda}_\alpha)
$$

is always nonnegative for any arbitrary $\ddot{\lambda}_\alpha^*$, $\ddot{u}_i^*$ satisfying conditions (96). Assuming position (71), the above difference becomes

$$
\begin{aligned}
\Delta\Psi^{\mathrm{L}} = \frac{1}{2}\int_{V_0}\rho_0\Delta\ddot{u}_i\Delta\ddot{u}_i\,\mathrm{d}V_0+\int_{V_0}\rho_0\ddot{u}_i\Delta\ddot{u}_i\,\mathrm{d}V_0-\int_{V_0}\rho_0F_{0i}\Delta\ddot{u}_i\,\mathrm{d}V_0\\
-\int_{S_{T0}}T_{0i}\Delta\ddot{u}_i\,\mathrm{d}S_{T0}+\sum_\alpha\int_{V_0}k_\alpha\Delta\ddot{\lambda}_\alpha\frac{\rho_0}{\rho}\,\mathrm{d}V_0.
\end{aligned}
\tag{97}
$$

Using the equilibrium equations, eqns (52), and Gauss's theorem we obtain

$$\int_{V_0} \rho_0 \ddot{u}_i \Delta\ddot{u}_i \,\mathrm{d}V_0 = \int_{V_0} \left(S_{jk}\frac{\partial x_i}{\partial a_k}\right)_{,j} \Delta\ddot{u}_i \,\mathrm{d}V_0 + \int_{V_0} \rho_0 F_{0i} \Delta\ddot{u}_i \,\mathrm{d}V_0$$

$$= \int_{S_{T0}} T_{0i}\Delta\ddot{u}_i \,\mathrm{d}S_{T0} - \int_{V_0} S_{jk}\frac{\partial x_i}{\partial a_k}\Delta\ddot{u}_{i,j}\,\mathrm{d}V_0 + \int_{V_0} \rho_0 F_{0i}\Delta\ddot{u}_i\,\mathrm{d}V_0 \qquad (98)$$

and then the difference $\Delta\Psi^{\mathrm{L}}$ reduces

$$\Delta\Psi^{\mathrm{L}} = \frac{1}{2}\int_{V_0} \rho_0 \Delta\ddot{u}_i \Delta\ddot{u}_i \,\mathrm{d}V_0 - \int_{V_0} S_{jk}\frac{\partial x_i}{\partial a_k}\Delta\ddot{u}_{i,j}\,\mathrm{d}V_0 + \sum_\alpha \int_{V_0} k_\alpha \Delta\ddot{\lambda}_\alpha \frac{\rho_0}{\rho}\,\mathrm{d}V_0. \qquad (99)$$

The symmetry of the Kirchhoff stress tensor S_{jk}, the use of eqns (A3) and $(53)_3$ make it possible to write the second integral of eqn (99) in the form

$$-\int_{V_0} S_{jk}\left(\frac{\partial x_i}{\partial a_k}\Delta\ddot{u}_{i,j}\right)\mathrm{d}V_0 = -\frac{1}{2}\int_{V_0} S_{jk}\left[\frac{\partial x_i}{\partial a_k}\Delta\ddot{u}_{i,j} + \frac{\partial x_i}{\partial a_j}\Delta\ddot{u}_{i,k}\right]\mathrm{d}V_0$$

$$= -\frac{1}{2}\int_{V_0} S_{jk}[\Delta\ddot{u}_{k,j} + \Delta\ddot{u}_{i,j}u_{i,k} + \Delta\ddot{u}_{j,k} + \Delta u_{i,k}u_{i,j}]\,\mathrm{d}V_0$$

$$= -\int_{V_0} S_{jk}\Delta\ddot{E}_{jk}\,\mathrm{d}V_0 \qquad (100)$$

then

$$\Delta\Psi^{\mathrm{L}} = \frac{1}{2}\int_{V_0} \rho_0 \Delta\ddot{u}_i \Delta\ddot{u}_i \,\mathrm{d}V_0 - \int_{V_0} S_{jk}\Delta\ddot{E}_{jk}\,\mathrm{d}V_0 + \sum_\alpha \int_{V_0} k_\alpha \Delta\ddot{\lambda}_\alpha \frac{\rho_0}{\rho}\,\mathrm{d}V_0 \qquad (101)$$

which using condition $(96)_2$ is transformed into

$$\Delta\Psi^{\mathrm{L}} = \frac{1}{2}\int_{V_0} \rho_0 \Delta\ddot{u}_i \Delta\ddot{u}_i \,\mathrm{d}V_0 - \sum_\alpha \int_{V_0} (N^S_{\alpha jk}S_{jk} - k_\alpha)\Delta\ddot{\lambda}_\alpha \frac{\rho_0}{\rho}\,\mathrm{d}V_0. \qquad (102)$$

The first integral of eqn (102) is always positive and equal to zero if and only if

$$\ddot{u}^*_j = \ddot{u}_j \quad \text{in } V_0. \qquad (103)$$

The second integral, in the region V_{P0}, vanishes because $\varphi_\alpha = 0$; in the remaining region V_{R0}, by virtue of the condition

$$\varphi_\alpha \ddot{\lambda}_\alpha = (N^S_{\alpha ij}S_{ij} - k_\alpha)\ddot{\lambda}_\alpha = 0$$

the term reduces to

$$-\sum_\alpha \int_{V_0} (N^S_{\alpha jk}S_{jk} - k_\alpha)\Delta\ddot{\lambda}_\alpha \frac{\rho_0}{\rho}\,\mathrm{d}V_0 = -\sum_\alpha \int_{V_0} (N^S_{\alpha jk}S_{jk} - k_\alpha)\ddot{\lambda}^*_\alpha \frac{\rho_0}{\rho}\,\mathrm{d}V_0 \qquad (104)$$

which is always nonnegative for every non-vanishing $\ddot{\lambda}^*_\alpha$ as a consequence of condition $(96)_3$ and because $\varphi_\alpha \leqslant 0$, and is zero if and only if

$$\ddot{\lambda}^*_\alpha = 0 \quad \text{where} \quad N^S_{\alpha ij}S^*_{ij} - k_\alpha < 0 \quad \text{in } V_{R0}. \qquad (105)$$

This proves that

$$\Psi^{L}(\ddot{u}_i^*, \ddot{\lambda}_\alpha^*) \geq \Psi^{L}(\ddot{u}_i, \ddot{\lambda}_\alpha)$$

for every acceleration $\ddot{u}_i^*$ and plastic multiplier acceleration $\ddot{\lambda}_\alpha^*$ satisfying conditions (96), the equality sign holding if and only if eqns (103) and (105) are satisfied.

This proves the theorem. Again it follows from this proof that the actual acceleration field $\ddot{u}_i$ is univocally defined; on the contrary, the actual plastic multiplier accelerations $\ddot{\lambda}_\alpha$ may not be determined (even for non-vanishing hardening coefficients $H_{\alpha\beta}$) because the only limitations that $\ddot{\lambda}_\alpha^*$ must satisfy are

$$\ddot{E}_{ij}^* = \sum_\alpha \ddot{\lambda}_\alpha^* \left(\frac{\rho_0}{\rho} N_{\alpha ij}^S\right) + \sum_\alpha \dot{\lambda}_\alpha \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho_0}{\rho} N_{\alpha ij}^S\right)$$

$$\ddot{\lambda}_\alpha^* \leq 0 \quad \text{if} \quad N_{\alpha ij}^S S_{ij} - k_\alpha = 0 \quad \text{in } V_{R0}$$

$$\ddot{\lambda}_\alpha^* = 0 \quad \text{if} \quad N_{\alpha ij}^S S_{ij} - k_\alpha < 0 \quad \text{in } V_{R0} \tag{106}$$

which may not always be sufficient to determine S_{ij} univocally.

For this theorem too, finally it is easy to show that the statement of the theorem could have been obtained starting from the corresponding Eulerian theorem formulation (95), (96) and using the formal relationship (see Appendix) between the variables of the Eulerian and Lagrangian approach.

5. LINKS WITH PREVIOUS THEOREMS

The theorem of Stolarsky and Belytschko[3] and the less recent theorems of Capurso[2] appear to be different forms of Theorem II and special cases of both theorems, respectively.

Stolarsky and Belytschko's theorem corresponds to Theorem II of the Eulerian approach when the functional dependence from the variable $\ddot{\lambda}_\alpha^*$ and the relevant conditions are implicitly expressed in the functional. This may be obtained by defining as kinematically admissible the fields σ_{ij}^*, $\ddot{u}_i^*$ satisfying the following conditions:

(a) $\ddot{u}_j^*$ satisfies the boundary conditions (eqn $(69)_1$);

(b) σ_{ij} are consistent with the relevant field D_{ij}, eqn (26) in V_P. In V_R they correspond to D_{ij}^{0*} through eqn (28) where D_{ij}^{0*} is given by eqn (18) $\ddot{u}_j$ being replaced by $\ddot{u}_j^*$ and respecting limitations (29) with $\ddot{\lambda}_\alpha = \ddot{\lambda}_\alpha^* \geq 0$.

In other words, the stresses $\sigma_{ij}^*(\ddot{u}_j^*)$ corresponding to kinematically admissible accelerations $\ddot{u}_i^*$ are defined both in the region V_P and (although not univocally) in the region V_R, by the conditions

$$D_{ij}^{0*}(\ddot{u}_i^*) = N_{\alpha ij}^\sigma \ddot{\lambda}_\alpha^*$$

$$N_{\alpha ij}^\sigma \sigma_{ij}^*(\ddot{u}_i^*) - k_\alpha = 0 \quad \text{if} \quad \ddot{\lambda}_\alpha^* > 0$$

$$N_{\alpha ij}^\sigma \sigma_{ij}^*(\ddot{u}_i^*) - k_\alpha \leq 0 \quad \text{if} \quad \ddot{\lambda}_\alpha^* = 0 \tag{107}$$

where the last two equations can be more concisely expressed as

$$(N_{\alpha ij}^\sigma \sigma_{ij}^* - k_\alpha)\ddot{\lambda}_\alpha^* = 0. \tag{108}$$

Then, bases on eqns (68) and (69) and using eqns (108) and $(107)_1$ Theorem II can be written as follows.

Among all kinematically admissible fields of accelerations $\ddot{u}_j^*$, the solution $\ddot{u}_i$ minimizes the functional

$$J(\ddot{u}_j^*) = \frac{1}{2}\int_V \rho \ddot{u}_j^* \ddot{u}_j^* \,\mathrm{d}V - \int_V \rho F_j \ddot{u}_j^* \,\mathrm{d}V - \int_{S_T} T_j \ddot{u}_j^* \,\mathrm{d}S_T + \int_V \sigma_{ij}^*(\ddot{u}_j^*) D_{ij}^{0*}(\ddot{u}_j^*) \,\mathrm{d}V. \tag{109}$$

This is Stolarsky and Belytschko's theorem[3], which, however, in the last integral of eqn

(109), loses the quadratic form of the functional, typical of the above Theorems I and II, when the yield surface is generally convex (not necessarily piecewise linear) and $N^{\sigma}_{\alpha ij}$ and k_α are suitable functions of σ_{ij}.

Finally, the particularization of Theorems I and II of the Lagrangian approach to the case of small strain–finite displacement theory quite naturally leads to Capurso's theorem[2] when considering that the new configuration of an original infinitesimal element must coincide with the initial one, except for a rigid body motion of any amplitude which has taken place, i.e.

$$\begin{aligned} &\rho \simeq \rho_0, \quad \mathrm{d}V \simeq \mathrm{d}V_0 \\ &\frac{\partial a_i}{\partial x_k} \simeq \frac{\partial x_k}{\partial a_i} = F_{ki} \\ &F_{ij} = R_{ih}U_{hj} \simeq R_{ij} \\ &\dot{E}_{ij} = F_{mi}D_{mn}F_{nj} \simeq R_{mi}D_{mn}R_{nj}. \end{aligned} \tag{110}$$

Besides, in eqn (42) the last term of the second member vanishes as can be shown using the equivalent expression (37′), taking eqns (110) and (8) into account, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial x_h}{\partial a_r}\frac{\partial x_k}{\partial a_j} R_{hr}R_{ks}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(R_{hi}R_{kj}R_{hr}R_{ks}) = \frac{\mathrm{d}}{\mathrm{d}t}(\delta_{ir}\delta_{js}) = 0 \tag{111}$$

and, then, eqns (40) and (42) of the constitutive laws, reduce to

$$\begin{aligned} &\dot{E}_{ij} = \sum_\alpha \dot{\lambda}_\alpha N^S_{\alpha ij} \\ &\ddot{E}_{ij} = \sum_\alpha \ddot{\lambda}_\alpha N^S_{\alpha ij}. \end{aligned} \tag{112}$$

Using the above positions, Theorems I and II of the Lagrangian approach come to coincide with Capurso's small-strain, large-displacement theorems.

In particular functional (81) written in the form

$$\Phi^L(S^*_{ij}, \ddot{u}^*_i) = \frac{1}{2}\int_{V_0} \rho_0 \ddot{u}^*_i \ddot{u}^*_i \,\mathrm{d}V_0 + \frac{1}{2}\int_{V_0} S^*_{ij}(A_{ij} - 2\dot{u}_{r/i}\dot{u}_{r/j})\,\mathrm{d}V_0 - \int_{S_{u0}} x_{k/j} S^*_{ij} n_{0i} \ddot{U}_k \,\mathrm{d}S_{u0} \tag{113}$$

transforms itself, taking into account eqn (90′), into

$$\Phi_c(S^*_{ij}, \ddot{u}^*_i) = \frac{1}{2}\int_{V_0} \rho_0 \ddot{u}^*_i \ddot{u}^*_i \,\mathrm{d}V_0 - \int_{V_0} S^*_{ij}\dot{u}_{r/i}\dot{u}_{r/j}\,\mathrm{d}V_0 - \int_{S_{u0}} x_{k/j} S^*_{ij} n_{0i} \ddot{U}_k \,\mathrm{d}S_{u0} \tag{114}$$

which is the functional of Theorem I of Capurso[2].

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APPENDIX

The above widely used relationships between static, kinematic and material behaviour quantities relevant to the Eulerian and Lagrangian approach are summarized here.

Denoting by x_i and a_i the current (at time t) and the original (at time $t = 0$) positions of the same material particle, the Eulerian and Lagrangian descriptions of the geometry are based on the relations[4,5]

$$a_i = x_i(t) - u_i[x_i(t), t] \quad \text{(Eulerian)} \tag{A1}$$

$$x_i(t) = a_i + u_i(a_i, t) \quad \text{(Lagrangian)} \tag{A2}$$

implying (if $\partial/\partial a_j$ is denoted by $/\bar{j}$) the following relations:

$$\frac{\partial a_i}{\partial x_j} = \delta_{ij} - \frac{\partial u_i}{\partial x_j} = \delta_{ij} - u_{i/j}$$
$$\frac{\partial x_i}{\partial a_j} = \delta_{ij} + \frac{\partial u_i}{\partial a_j} = \delta_{ij} + u_{i/\bar{j}} \tag{A3}$$

and the following material time derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_i) = \dot{u}_i = \frac{\partial u_i}{\partial t} + u_{i/k}\dot{u}_k \tag{A4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u_i}{\partial x_j}\right) = \dot{u}_{i/j} - \dot{u}_{k/j}u_{i/k}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial u_i}{\partial a_j}\right) = \frac{\partial \dot{u}_i}{\partial a_j} \tag{A5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{u}_i}{\partial x_j}\right) = \ddot{u}_{i/j} - \dot{u}_{k/j}\dot{u}_{i/k}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \dot{u}_i}{\partial a_j}\right) = \frac{\partial \ddot{u}_i}{\partial a_j}. \tag{A6}$$

Deformations

Almansi e_{ij} and Green E_{ij} tensors as well as the deformation gradient F_{ij} are given and correlated by the following relations:

$$e_{ij} = \tfrac{1}{2}(u_{i/j} + u_{j/i} - u_{k/i}u_{k/j}) \tag{A7}$$

$$E_{ij} = \tfrac{1}{2}(u_{i/\bar{j}} + u_{j/\bar{i}} + u_{k/\bar{i}}u_{k/\bar{j}}) \tag{A8}$$

$$F_{ij} = \frac{\partial x_i}{\partial a_j} = \delta_{ij} + \frac{\partial u_i}{\partial a_j} \tag{A9}$$

$$e_{ij} = E_{hk}\frac{\partial a_h}{\partial x_i}\frac{\partial a_k}{\partial x_j}, \quad e_{ij} = \frac{1}{2}\left(F_{jk} - \frac{\partial a_k}{\partial x_j}\right)\frac{\partial a_k}{\partial x_i} \tag{A10}$$

$$E_{ij} = e_{hk}\frac{\partial x_h}{\partial a_i}\frac{\partial x_k}{\partial a_j}, \quad E_{ij} = \tfrac{1}{2}(F_{ki}F_{kj} - \delta_{ij}) \tag{A11}$$

$$F_{ki}\frac{\partial a_i}{\partial x_h} = 2e_{hk} + \frac{\partial a_i}{\partial x_k}\frac{\partial a_i}{\partial x_h}, \quad F_{ij} = 2E_{jh}\frac{\partial a_h}{\partial x_i} + \frac{\partial a_j}{\partial x_i}. \tag{A12}$$

Stress density and forces

The Cauchy $\sigma_{ij} = \sigma_{ji}$, Kirchhoff $S_{ij} = S_{ji}$ and Lagrange $T_{ij} \neq T_{ji}$ stress tensors are brought into relation by

$$\sigma_{ij} = \frac{\rho}{\rho_0}\frac{\partial x_i}{\partial a_h}\frac{\partial x_j}{\partial a_k}S_{hk}, \quad \sigma_{ij} = \frac{\rho}{\rho_0}\frac{\partial x_j}{\partial a_k}T_{ki} \tag{A13}$$

$$S_{ij} = \frac{\rho_0}{\rho}\frac{\partial a_i}{\partial x_h}\frac{\partial a_j}{\partial x_k}\sigma_{hk}, \quad S_{ij} = T_{ik}\frac{\partial a_j}{\partial x_k} \tag{A14}$$

$$T_{ij} = \frac{\rho_0}{\rho}\frac{\partial a_i}{\partial x_k}\sigma_{kj}, \quad T_{ij} = S_{ik}\frac{\partial x_j}{\partial a_k} \tag{A15}$$

while density and forces are related as follows:

$$\rho_0\,\mathrm{d}V_0 = \rho\,\mathrm{d}V,\quad F_{0i} = F_i,\quad T_{0i}\,\mathrm{d}S_0 = T_i\,\mathrm{d}S \tag{A16}$$

$$\frac{\partial a_i}{\partial x_k} n_{0i}\,\mathrm{d}S_0 = \frac{\rho}{\rho_0} n_k\,\mathrm{d}S \tag{A16'}$$

where "0" refers to the variable value at time $t = 0$.

Yield functions

Depending on the chosen stress space, the piecewise-linearized yield surface, is expressed as

$$\begin{aligned} \varphi_\alpha &= N^\sigma_{\alpha ij}\sigma_{ij} - k_\alpha \leqslant 0 \\ \varphi_\alpha &= N^S_{\alpha ij}S_{ij} - k_\alpha \leqslant 0 \\ \varphi_\alpha &= \hat{N}_{\alpha ij}\hat{\sigma}_{ij} - k_\alpha \leqslant 0 \end{aligned} \tag{A17}$$

where $\hat{\sigma}_{ij}$ are the components of the corotational stress tensor

$$\hat{\sigma}_{ij} = R_{hi}\sigma_{hk}R_{kj} \tag{A18}$$

R_{hi} being a generic rigid body motion.

The relations between the outward normal unit vectors $N_{\alpha ij}$ are

$$\hat{N}_{\alpha ij} = R_{hi}N^\sigma_{\alpha hk}R_{kj},\qquad \hat{N}_{\alpha ij} = \frac{\rho_0}{\rho}\frac{\partial a_r}{\partial x_h}\frac{\partial a_s}{\partial x_k}R_{hi}N^S_{\alpha rs}R_{kj} \tag{A19}$$

$$N^\sigma_{\alpha ij} = R_{ih}\hat{N}_{\alpha hk}R_{jk},\qquad N^\sigma_{\alpha ij} = \frac{\rho_0}{\rho}\frac{\partial a_h}{\partial x_i}\frac{\partial a_k}{\partial x_j}N^S_{\alpha hk} \tag{A20}$$

$$N^S_{\alpha ij} = \frac{\rho}{\rho_0}\frac{\partial x_h}{\partial a_i}\frac{\partial x_k}{\partial a_j}R_{hr}\hat{N}_{\alpha rs}R_{ks},\quad N^S_{\alpha ij} = \frac{\rho}{\rho_0}\frac{\partial x_h}{\partial a_i}\frac{\partial x_k}{\partial a_j}N^\sigma_{\alpha hk}. \tag{A21}$$